OPTIMISATION | KEY FACTS

- To find the minimum or maximum value of a function $\mathrm{f}(x)$:
  - Solve $\mathrm{f}'(x) = 0$
  - Use $\mathrm{f}''(x)$ to determine if it is a minimum or maximum point.

OPTIMISATION | EXAMPLE-PROBLEM PAIRS

1E. A wire of length 12 cm is bent to form a rectangle.

(a) Show that the area, A, is given by $A = 6x - x^2$, where x is the width of the rectangle.

$P = 2x + 2y$

$\therefore\ 12 = 2x + 2y$

$6 = x + y$

$y = 6 - x$

$A = xy$

$= x(6 - x)$

$= 6x - x^2$

(b) Find the maximum possible area.

$\frac{dA}{dx} = 6 - 2x$

Min/Max occurs when $\frac{dA}{dx} = 0$

$\therefore\ 0 = 6 - 2x$

$x = 3$

FIND ANY STATIONARY POINTS

$\frac{d^2A}{dx^2} = -2 < 0$

DETERMINE IF THEY ARE MINIMA/MAXIMA

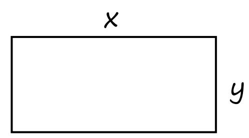
$\therefore$ Maximum area when $x = 3$

$\therefore$ Maximum area $= 6(3) - (3)^2$

$= 9 \text{ cm}^2$

1P. A farmer wants to build a rectangular sheep pen with length x and width y. She has $20\,\text{m}$ of fencing in total and wants the area inside the pen to be as large as possible.

(a) Show that the area of the pen is given by $A = 10x - x^2$,



(b) Find the length and width of the pen that give the maximum area.

2E. An open-topped cylindrical can has base radius r and height h. The external surface area $A = 243\pi \text{ cm}^2$. Show that the volume of the can is given by

$$V = \frac{\pi}{2}(243r - r^3)$$

$$V = \pi r^2 h$$
$$A = \pi r^2 + 2\pi rh$$

Since $A = 243\pi$,

$$243\pi = \pi r^2 + 2\pi rh$$
$$243 = r^2 + 2rh$$
$$h = \frac{243 - r^2}{2r}$$
$$\therefore \quad V = \pi r^2\left(\frac{243 - r^2}{2r}\right)$$
$$= \frac{\pi}{2}\left(243r - r^3\right)$$

2E. (b) Hence find the maximum capacity of the can.

$$V = \frac{\pi}{2}\left(243r - r^3\right)$$
$$V = \frac{243\pi}{2}r - \frac{\pi}{2}r^3$$

WRITE IN A FORM WE CAN DIFFERENTIATE

$$\frac{dV}{dr} = \frac{243\pi}{2} - \frac{3\pi}{2}r^2$$

Min/Max occurs when $\frac{dV}{dr} = 0$

$$\therefore \quad 0 = \frac{243\pi}{2} - \frac{3\pi}{2}r^2$$
$$\frac{3\pi}{2}r^2 = \frac{243\pi}{2}$$
$$r^2 = 81$$
$$r = \pm 9$$

WE CANNOT HAVE A NEGATIVE RADIUS

Since $r > 0$, $r = 9$.

$$\frac{d^2V}{dr^2} = -3\pi r$$
$$\left.\frac{d^2V}{dr^2}\right|_{r=9} = -27\pi < 0$$

$\therefore$ $r = 9$ is a maximum point.

$$\therefore \quad V_{max} = \frac{\pi}{2}\left(243 \times 9 - (9)^3\right)$$
$$= 729\pi \text{ cm}^3$$

An open-topped cylindrical pie tin is t cm high with a radius of r cm. The volume of the pie tin is 1000 cm³.

(a) Show that the surface area of the tin is given by

$$A = \pi r^2 + \frac{2000}{r}$$

(b) Find the minimum surface area.