## EXPONENTIAL GROWTH AND DECAY | KEY FACTS

$$N = N_0 a^{kt}$$

- $N$ Current quantity
- $k$ Proportionality constant
- $t$ Current time
- $N_0$ Initial quantity (at $t = 0$)

+ve Power ⇒ exponential **growth**

-ve Power ⇒ exponential **decay**

- A quantity grows or decays exponentially when the rate of change is proportional to the quantity itself.
- Mathematically, this is written as $\frac{\mathrm{d}N}{\mathrm{d}t} \propto N$ or $\frac{\mathrm{d}N}{\mathrm{d}t} = kN$

## EXPONENTIAL GROWTH AND DECAY | EXAMPLE-PROBLEM PAIRS

1E. A simple model of a population of bacteria states that the number of bacteria, $N$, grows exponentially, such that

$$N = Ae^{kt},$$

where $t$ is time, in minutes, since the start of the experiment.

Initially, there were 2000 bacteria and after 5 minutes, this number has grown to 7000.

(a) Find the values of the constants $A$ and $k$.

(b) According to this model, how many bacteria will there be in the dish after another 5 minutes?

1P. A nonlethal disease is spreading at a rate which can be modelled exponentially, such that

$$N = Be^{\lambda t},$$

where $N$ represents the number of infected people at a given time, $t$ hours, since the outbreak was first identified.

Initially, there are 200 people infected with the disease. After 10 hours, there are 320 people infected.

(a) Find the values of the constants $B$ and $\lambda$

(b) According to this model, approximately how many people will be infected 72 hours after the outbreak was first identified?

(a) When $t = 0$, $N = 2000$.

$\therefore\ 2000 = Ae^0$

$2000 = A$

$\therefore\ N = 2000e^{kt}$

When $t = 5$, $N = 7000$.

$7000 = 2000e^{5k}$

$3.5 = e^{5k}$

$\ln 3.5 = 5k$

$k = \frac{1}{5}\ln 3.5 \quad (= 0.250\ldots)$

$N = 2000e^{(0.250\ldots)t}$

(b) When $t = 10$ (after another 5 mins)

$N = 2000e^{(0.250\ldots)\times 10}$

$= 24500$ (3 s.f.)

## EXPONENTIAL GROWTH AND DECAY | EXAMPLE-PROBLEM PAIRS

2E. The temperature $T$ °C of a cup of coffee after t minutes is given by $T = 20 + 60e^{-0.1t}$.

(a) What is the initial temperature of the coffee.

(b) What is the temperature of the coffee after 5 minutes?.

(c) After how long is the temperature of the coffee 25°C?

(d) What is the temperature of the room?

(a) When $t = 0$,

$T = 20 + 60e^{0} = 80$ °C

(b) When $t = 5$,

$T = 20 + 60e^{-0.1\times5} = 56.4$ °C

(c) When $T = 25$ °C,

$25 = 20 + 60e^{-0.1t}$

$5 = 60e^{-0.1t}$

$\frac{1}{12} = e^{-0.1t}$

$\ln\left(\frac{1}{12}\right) = -0.1t$

$t = 24.8$ mins

(d) Over time the coffee will approach room temperature.

As $t \to \infty$, $60e^{-0.1\times5t} \to 0$

$\therefore\ T \to 20$ °C

Room temperature is 20°C

2P. The temperature $T$ °C of the water in a kettle $t$ minutes after boiling is given by the equation $T = 20 + 80e^{-0.5t}$.

(a) What is the initial temperature of the water?

(b) What is the temperature of the coffee after 8 minutes?

(c) After how long is the temperature of the coffee 30°C?

## RATES OF CHANGE | KEY FACTS

- To find the 'rate of change', we need to differentiate the original function. This must be done before substituting in a value for $t$.

## RATES OF CHANGE | EXAMPLE-PROBLEM PAIR S

3E. A population of flies grows exponentially, so that its size can be modelled by the equation $N = Ae^{kt}$, where $N$ is the number of flies after $t$ weeks. At the time $t = 0$, the population size is $2400$ and it is increasing at a rate of $80$ flies per week.

Find the values of $A$ and $k$.

When $t = 0$, $N = 2400$,

$\therefore\ A = 2400$

$N = 2400e^{kt}$

$\frac{dN}{dt} = 2400ke^{kt}$

At $t = 0$, $\frac{dN}{dt} = 80$,

$\therefore\ 80 = 2400ke^{0}$

$k = \frac{1}{30}$

3P. The mass, in grams, of a substance in a chemical reaction is modelled by the equation $m = Ae^{-1.2t}$, where $t$ seconds is the time since the start of the reaction.

The initial mass of the substance was $72$g.

(a) State the value of $A$.

(b) Find the rate at which the mass is decreasing $5$ seconds after the start of the reaction.

4E. The value £$V$ of a car is given by the formula $V = A \times 2^{-kt}$ where $t$ is the age of the car in years.

On 1st January 2012, the car is valued at £14 200.
On 1st January 2015 it is valued at £9600.

(a) Find the values of the constants A and $k$.

(b) Find the year in which the car will be valued at less than £5800.

(a) When $t = 0$, $V = 14200$,

$$\therefore \quad A = 14200$$

$$\therefore \quad V = 14200 \times 2^{-kt}$$

When $t = 3$, $V = 9600$,

$$\therefore \quad 9600 = 14200 \times 2^{-3k}$$

$$\frac{48}{71} = 2^{-3k}$$

$$\log_2\left(\frac{48}{71}\right) = -3k$$

$$k = -\frac{1}{3}\log_2\left(\frac{48}{71}\right) = 0.188\ldots$$

(b)

$$V = 14200 \times 2^{-0.188t}$$

$$5800 > 14200 \times 2^{-0.188t}$$

$$\frac{29}{71} > 2^{-0.188t}$$

$$\ln\left(\frac{29}{71}\right) > \ln 2^{-0.188t}$$

$$\ln\left(\frac{29}{71}\right) > -0.188t \ln 2$$

$$\frac{\ln\left(\frac{29}{71}\right)}{-0.188\ln 2} > t$$

We are dividing by a negative, so we flip the inequality sign.

$$6.87 > t$$

$$2012 + 6.87 = 2018.87$$

The year is 2018

4P. The value £$V$ of an initial investment £$P$ at the end of $n$ years is given by the formula

$$V = P\left(1 + \frac{r}{100}\right)^n,$$

where $r$% is the fixed interest rate.

£3000 is invested at a fixed interest rate of 4%.

Assuming that the money was invested on January 1st 2017, find the year in which the value of the investment will exceed £14 000.

TIP:

Taking logs of both sides with a base between 0 and 1 causes inequality signs to be flipped. To avoid potential mistakes, it is usually wise to take natural logs when inequalities are involved (see example 4E part b).

## FURTHER EXAMPLES | EXAMPLE-PROBLEM PAIRS

5E. The number of radioactive atoms in a sample of radioactive material is decaying exponentially. It is modelled by the equation $N = N_0e^{-0.08t}$ , where $N$ is the remaining number of radioactive atoms of the substance $t$ years after the start of the reaction.

Find the time at which the number of radioactive atoms remaining is half of the original number of radioactive atoms.

$N = N_0e^{-0.08t}$

Half of the atoms remain when $N = \frac{1}{2}N_0$

$\therefore \quad \frac{1}{2}N_0 = N_0e^{-0.08t}$

$\frac{1}{2} = e^{-0.08t}$

$\ln\left(\frac{1}{2}\right) = -0.08t$

$t = 8.66$ years

5P. An antiviral drug is being used to treat an infection. The number of remaining infected cells is given by $N = N_0e^{-0.5t}$ where $N$ is the number of infected cells $t$ days after the drug is first administered.

Find the minimum number of full days required for the number of infected cells to fall to less than 10% of the original number of infected cells.

## EXTENSION | EXAMPLE-PROBLEM PAIRS

The height of a particular species of plant is increasing exponentially, such that $h = 20 - 20e^{-t/5}$, where $h$ is the height of the plant $t$ days after it is planted.

Show that $\frac{dh}{dt} = 4 - \frac{1}{5}h$.

A cup of tea cools exponentially. The temperature, $T$ °C after $t$ minutes is given by $T = 20 + 75e^{-t/15}$.

Show that $\frac{dT}{dt} = \frac{1}{15}(20 - T)$.

TIP:

It is usually easier to spot how to write the answer in the correct form if you do not simplify after differentiating.

It is often helpful to rearrange the original equation **before** differentiating.

## REDUCTION TO LINEAR FORM | KEY FACTS

- We can reduce exponential and polynomial equations to linear form if we use a logarithmic scale.
  - $y = kx^n \rightarrow \log y = \log k + n \log x$ (plot $\log y$ against $\log x$)
  - $y = ka^x \rightarrow \log y = \log k + x \log a$ (plot $\log y$ against $x$)

## REDUCTION TO LINEAR FORM | EXAMPLE-PROBLEM PAIRS

The relationship between two variables $x$ and $y$ is believed to be of the form $y = kx^n$, where $k$ and $n$ are constants.
In an experiment, the values of $x$ and $y$ are recorded (see table below)
Verify that the model $y = kx^n$ is appropriate and find the approximate values of the constants $k$ and $n$.

$y = kx^n$

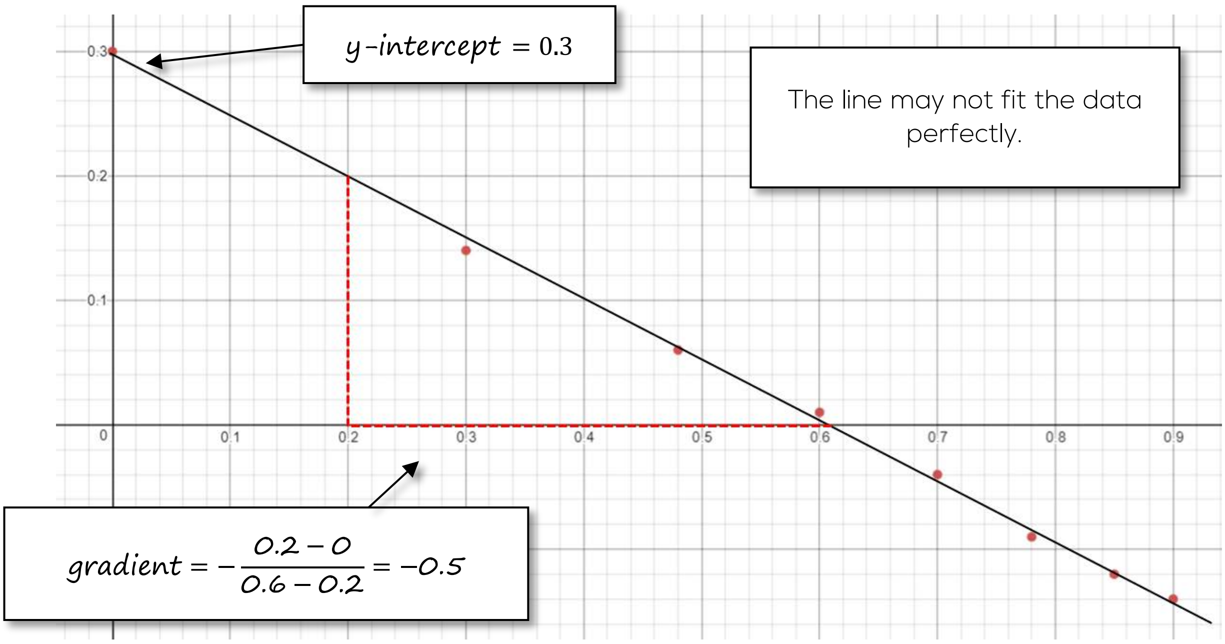
$\log y = \log(kx^n)$

$\log y = \log k + \log x^n$

$\log y = \log k + n \log x$

Intercept ($\log k$) — Gradient (n)

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 1.98 | 1.39 | 1.16 | 1.01 | 0.91 | 0.82 | 0.75 | 0.72 |
| $\log x$ | 0 | 0.30 | 0.48 | 0.60 | 0.70 | 0.78 | 0.85 | 0.90 |
| $\log y$ | 0.30 | 0.14 | 0.06 | 0.01 | -0.04 | -0.09 | -0.12 | -0.14 |




$y\text{-intercept} = 0.3$

$\therefore \log k = 0.3$

$k = 10^{0.3}$

$= 2.0$ (2 s.f.)

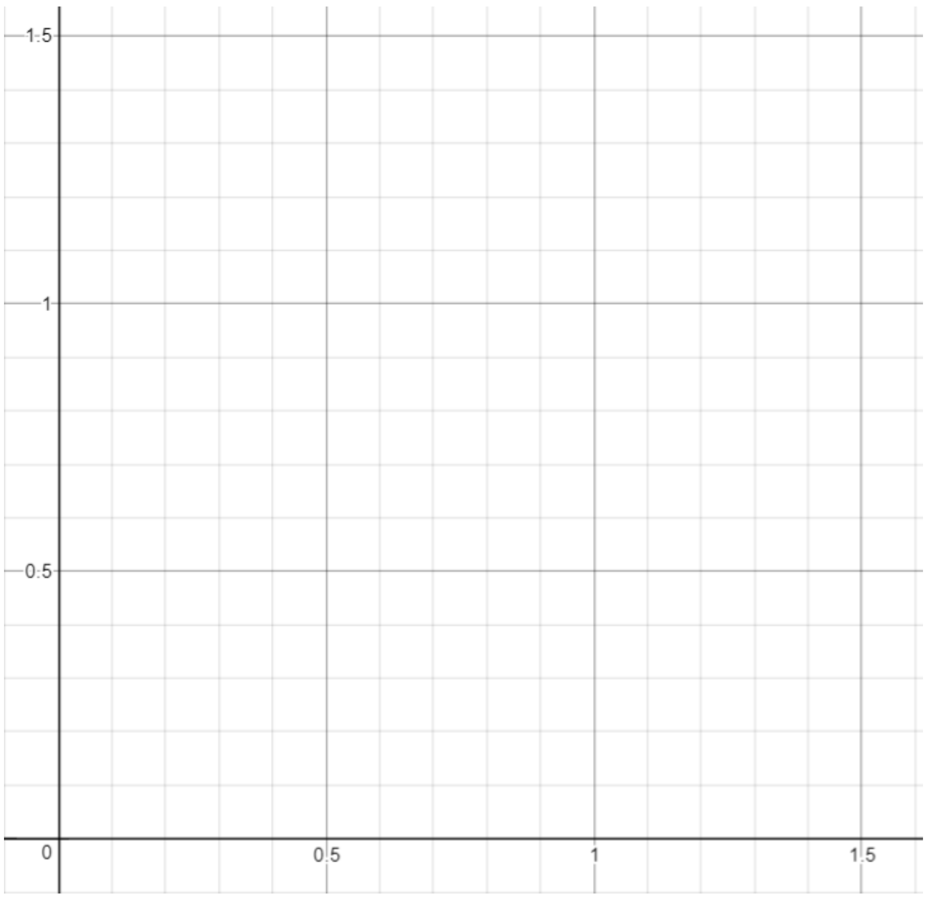
$\text{gradient} = -0.5$

$\therefore n = -0.5$

$$\therefore y = 2.0x^{-0.5} \quad \left(\text{or} \quad y = \frac{2.0}{\sqrt{x}}\right)$$

The relationship between two variables $t$ and $p$ is believed to be of the form $p = at^b$, where $a$ and $b$ are constants.
In an experiment, the values of $t$ and $p$ are recorded (see table below)
Verify that the model $p = at^b$ is appropriate and find the approximate values of the constants $a$ and $b$.

| $t$ | 2 | 5 | 8 | 13 | 25 |
|---|---|---|---|---|---|
| $p$ | 3 | 7 | 10 | 16 | 29 |
| | | | | | |
| | | | | | |



The relationship between two variables $x$ and $y$ is believed to be of the form $y = ka^x$, where $k$ and $a$ are constants.

In an experiment, the values of $x$ and $y$ are recorded (see table below)

Verify that the model $y = ka^x$ is appropriate and find the approximate values of the constants $k$ and $a$.

$$y = ka^x$$

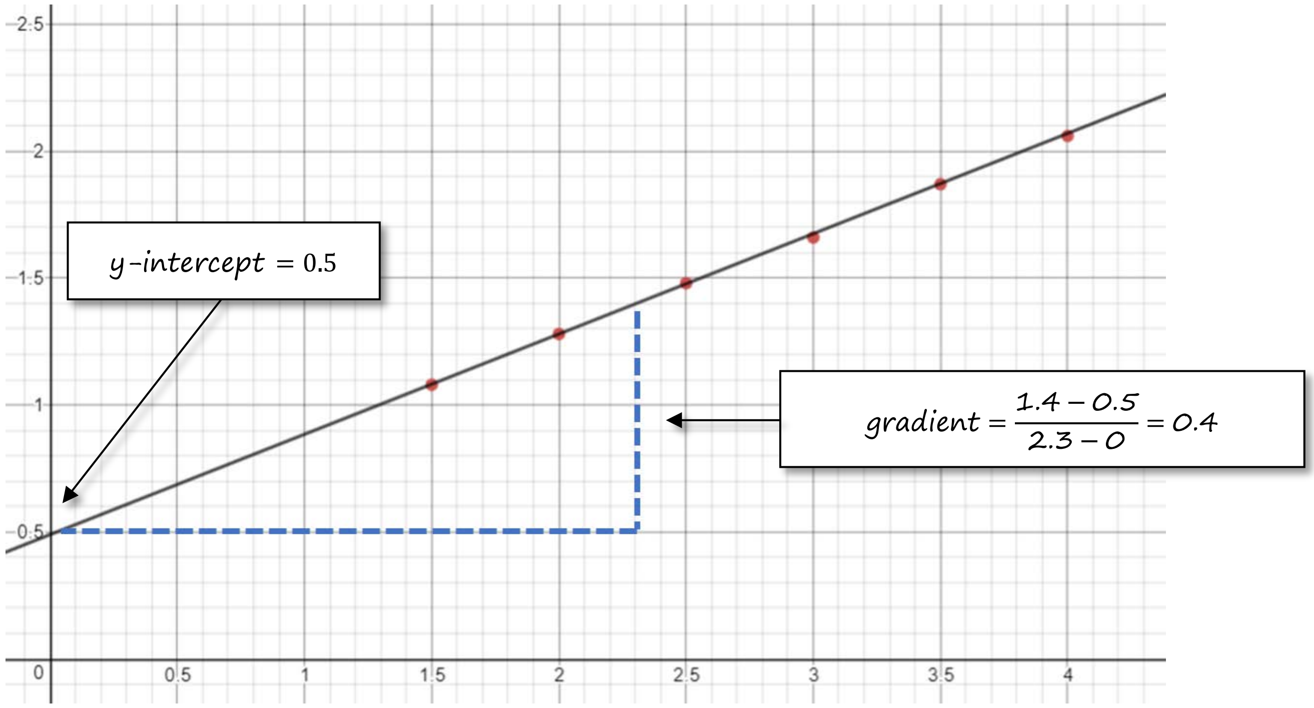
$$\log y = \log(ka^x)$$

$$\log y = \log k + \log a^x$$

$$\log y = \log k + x \log a$$

Intercept ($\log k$) Gradient ($\log a$)

| $x$ | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|---|
| $y$ | 12 | 19 | 30 | 46 | 74 | 116 |
| $\log y$ | 1.08 | 1.28 | 1.48 | 1.66 | 1.87 | 2.06 |




y-intercept $= 0.5$

$\therefore \quad \log k = 0.5$

$k = 10^{0.5}$

$= 3.2$ (2 s.f.)

gradient $= 0.4$

$\therefore \quad \log a = 0.4$

$a = 10^{0.4}$

$= 2.5$

$$\therefore \quad y = 3.2 \times 2.5^x$$

The relationship between two variables $p$ and $q$ is believed to be of the form $q = ab^p$, where $a$ and $b$ are constants.

In an experiment, the following values of $p$ and $q$ are recorded.

| $p$ | 1.2 | 3.4 | 5.7 | 6.2 | 7.4 | 9.8 |
|---|---|---|---|---|---|---|
| $q$ | 2.5 | 3.7 | 5.8 | 6.1 | 7.7 | 11.9 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |

(i) Plot the graph of log $q$ against $p$, and explain why this tells you that the model $q = ab^p$ is appropriate.

(ii) Use your graph to estimate the values of $a$ and $b$.

(iii) Estimate the value of $q$ when $p = 12$.