EXPONENTIAL GROWTH AND DECAY | KEY FACTS

$$N = Ae^{kt}$$

- N Current quantity
- k Proportionality constant
- t Current time
- A Initial quantity (at $t = 0$)

+ve Power ⇒ exponential **growth**

-ve Power ⇒ exponential **decay**

- A quantity grows or decays exponentially when the **rate of change** is **proportional** to the **quantity itself**.
- Mathematically, this is written as $\boxed{\dfrac{\mathrm{d}N}{\mathrm{d}t} \propto N \quad \text{or} \quad \dfrac{\mathrm{d}N}{\mathrm{d}t} = kN}$

EXPONENTIAL GROWTH AND DECAY | EXAMPLE-PROBLEM PAIRS

1E. A simple model of a population of bacteria states that the number of bacteria (N thousand) grows exponentially, such that

$$N = Ae^{kt},$$

where t is time, in minutes, since the start of the experiment.

Initially, there were 2000 bacteria and after 5 minutes, this number has grown to 7000.

(a) Find the values of the constants A and k.

(b) According to this model, how many bacteria will there be in the dish after another 5 minutes?

1P. A nonlethal disease is spreading at a rate which can be modelled exponentially, such that

$$N = Be^{\lambda t},$$

where N represents the number of infected people at a given time, t hours, since the outbreak was first identified.

Initially, there are 200 people infected with the disease. After 10 hours, there are 320 people infected.

(a) Find the values of the constants B and λ

(b) According to this model, approximately how many people will be infected 72 hours after the outbreak was first identified?

(a) When $t = 0$, $N = 2000$.

$\therefore \; 2000 = Ae^0$

$2000 = A$

$\therefore \; N = 2000e^{kt}$

When $t = 5$, $N = 7000$.

$7000 = 2000e^{5k}$

$3.5 = e^{5k}$

$\ln 3.5 = 5k$

$k = \frac{1}{5}\ln 3.5 \quad (= 0.250\ldots)$

$N = 2000e^{(0.250\ldots)t}$

(b) When $t = 10$ (after another 5 mins)

$N = 2000e^{(0.250\ldots)\times 10}$

$= 24500$

Make sure to answer the question in context.

There are 24 500 bacteria in the dish.

EXPONENTIAL GROWTH AND DECAY | EXAMPLE-PROBLEM PAIRS

2E. The temperature T °C of a cup of coffee after t minutes is given by $T = 20 + 60e^{-0.1t}$.

(a) Find the initial temperature of the coffee.

(b) Find the temperature of the coffee after 5 minutes.

(c) After how long is the temperature of the coffee 25°C?

(d) Find the temperature of the room?

(a) When $t = 0$,

$$T = 20 + 60e^{0} = 80 \text{ °C}$$

(b) When $t = 5$,

$$T = 20 + 60e^{-0.1\times5} = 56.4 \text{ °C}$$

(c) When $T = 25$ °C

$$25 = 20 + 60e^{-0.1t}$$

$$5 = 60e^{-0.1t}$$

$$\tfrac{1}{12} = e^{-0.1t}$$

$$\ln\left(\tfrac{1}{12}\right) = -0.1t$$

$$t = 24.8 \text{ mins}$$

(d) Over time the coffee will approach room temperature.

As $t \to \infty$, $60e^{-0.1\times5t} \to 0$

$\therefore$ $T \to 20$ °C

Room temperature is 20°C

2P. The temperature T °C of the water in a kettle t minutes after boiling is given by the equation $T = 20 + 80e^{-0.5t}$.

(a) What is the initial temperature of the water?

(b) What is the temperature of the coffee after 8 minutes?

(c) After how long is the temperature of the coffee 30°C?

RATES OF CHANGE | EXAMPLE-PROBLEM PAIR S

3E. A population of flies grows exponentially, so that its size can be modelled by the equation $N = Ae^{kt}$, where N is the number of flies after t weeks. At the time $t = 0$, the population size is 2400 and it is increasing at a rate of 80 flies per week.

Find the values of A and k.

When $t = 0$, $N = 2400$,

$\therefore$ $A = 2400$

$$N = 2400e^{kt}$$

$$\frac{dN}{dt} = 2400ke^{kt}$$

At $t = 0$, $\frac{dN}{dt} = 80$,

$$\therefore\ 80 = 2400ke^{0}$$

$$k = \frac{1}{30}$$

3P. The mass, in grams, of a substance in a chemical reaction is modelled by the equation $m = Ae^{-1.2t}$, where t seconds is the time since the start of the reaction.

The initial mass of the substance was 72g.

(a) State the value of A.

(b) Find the rate at which the mass is decreasing 5 seconds after the start of the reaction.

The value £V of a car is given by the formula $V = A \times 2^{-kt}$ where t is the age of the car in years.

On 1st January 2012, the car is valued at £14 200.
On 1st January 2015 it is valued at £9600.

(a) Find the values of the constants A and k.
(hint: let January 2012 be $t = 0$)

(b) Find the year in which the car will be valued at less than £5800.

The value £V of an initial investment £P at the end of n years is given by the formula

$$V = P\left(1 + \frac{r}{100}\right)^n,$$

where r% is the fixed interest rate.

£3000 is invested at a fixed interest rate of 4%.

Assuming that the money was invested on January 1st 2017, find the year in which the value of the investment will exceed £14 000.

NOTES:

An equation of the form $A^{kt} = b$ can always be written in the form $\log_A b = kt$.

Special Case: Take care when taking logs with an inequality. If the base is less than 1, we must reverse the direction of the inequality.

e.g. $A < 0.5^{kt} \Leftrightarrow \log_{0.5} A > kt$

FURTHER EXAMPLES | EXAMPLE-PROBLEM PAIR 4

The number of radioactive atoms in a sample of radioactive material is decaying exponentially. It is modelled by the equation $N = N_0 e^{-0.08t}$, where N is the remaining number of radioactive atoms of the substance t years after the start of the reaction.

Find the time at which the number of radioactive atoms remaining is half of the original number of radioactive atoms.

An antiviral drug is being used to treat an infection. The number of remaining infected cells is given by $N = N_0 e^{-0.5t}$ where N is the number of infected cells t days after the drug is first administered.

Find the minimum number of full days required for the number of infected cells to fall to less than 10% of the original number of infected cells.

FURTHER EXAMPLES | EXAMPLE-PROBLEM PAIR 5

The height of a particular species of plant is increasing exponentially, such that $h = 20 - 20e^{-t/5}$, where h is the height of the plant t days after it is planted.

Show that $\frac{\mathrm{d}h}{\mathrm{d}t} = 4 - \frac{1}{5}h$.

A cup of tea cools exponentially. The temperature, T °C after t minutes is given by $T = 20 + 75e^{-t/15}$.

Show that $\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{15}(20 - T)$.

NOTES:

It is usually easier to spot how to write the answer in the correct form if you do not simplify after differentiating.

It is often helpful to rearrange the original equation **before** differentiating.